Discrete Electrical Network Approximations for the Solution to the Continuous Dirichlet Problem on the Unit Rectangle

Denote the unit rectangle by R. Suppose $\gamma\colon R \to \mathbb{R}^+$ is a conductivity function over the unit rectangle. Given a boundary function $\phi\colon \partial R \to \mathbb{R}$, there exists a unique electrical current function $u_\infty\colon R \to \mathbb{R}$ such that $u_\infty|_{\partial R} = \phi$ and which is γ-harmonic, i.e. $\nabla(\gamma \nabla u_\infty) = 0$. Additionally, u_∞ can be characterized as the minimizer over the γ-Dirchlet norm, $\|u\|_\gamma^2 = \int_R \gamma |\nabla u_\infty|^2 \mathrm{d}A$, over all functions $u\colon R \to \mathbb{R}$ with $u|_{\partial R} = \phi$.

Take an n by n grid of the unit rectangle, and divide each smaller square into two congruent triangles, whose shared leg is the diagonal of the square from the top left corner to the bottom right. Enumerating these triangles in some way, $R = \cup_{i=1}^{2n^2} T_i$. Let $S_n = \{u\colon R \to \mathbb{R} : u$ is continuous and differentiable except perhaps on ∂T_i and $\nabla u = 0$ where it exists.$\}$; in other words, $u \in S_n$ if and only if it is continuous and piecwise linear over the triangles of the n by n grid. Define a new conductivity function $\gamma_n\colon R \to \mathbb{R}^+$ by $\gamma_n(p) = \gamma_i$ if $p \in T_i$ where γ_i is the average of γ over T_i, and a new boundary function $\phi_n\colon \partial R \to \mathbb{R}^+$ where $\phi_n(p) = \phi(p)$ if p is an intersection point of the grid on the boundary (including the corners) and ϕ_n is linearly interpolated between these points. Now, note that with these new conductivity and boundary functions, we can identify the n by n grid with a discrete electrical lattice network. Each interior edge of the grid is the leg of two triangles T_i and T_j, to this edge in our discrete electrical network we associate the conductivity $\gamma_i + \gamma_j$. The boundary data of this network will be given by the values of ϕ_n at the boundary vertices. Define u_n to be the minimizer of the γ-Dirichlet norm over function $u \in S_n$ such that $u|_{\partial R} = \phi_n$. Such a minimizer exists and is unique because it is simply the solution of the Dirichlet problem for our discrete electrical network, as $\|u\|_\gamma^2 = \sum_{i=1}^{2n^2} \gamma_i |\nabla_{T_i} u|^2 |T_i|$ where $|T_i|$ is the area of our triangles and $\nabla_{T_i} u$ is the (constant) value of the gradient of u in T_i.

Our primary claim is that $\|u_n - u_\infty\|_\gamma \to 0$ as $n \to \infty$, i.e., the discrete γ-harmonic solutions of our electrical network approximate the smooth case in the limit.

$$
\begin{aligned}
\|u_n - u_\infty\|_\gamma^2 &= \int_R \gamma |\nabla(u_n - u_\infty)|^2 \mathrm{d}A \\
&= \int_R \gamma \nabla(u_n - u_\infty) \cdot \nabla(u_n - u_\infty) \mathrm{d}A \\
&= \int_R \gamma |\nabla u_\infty|^2 \mathrm{d}A - 2\int_R \gamma(\nabla u_\infty \cdot \nabla u_n)\mathrm{d}A + \int_R \gamma |\nabla u_n|^2 \mathrm{d}A \\
&= \|u_\infty\|_\gamma^2 + \|u_n\|_\gamma^2 - 2\langle u_\infty, u_n\rangle_\gamma
\end{aligned}
$$

Applying integration by parts, we see that

$$
\begin{aligned}
\langle u_\infty, u_n\rangle_\gamma &= \int_R (\gamma \nabla u_\infty) \cdot \nabla u_n \mathrm{d}A \\
&= \int_{\partial R} u_n(\gamma \nabla u_\infty \cdot \vec{n}) ds - \int_R u_n \nabla(\gamma \nabla u_\infty) \mathrm{d}A
\end{aligned}
$$

However, as u_∞ is γ-harmonic, $\nabla(\gamma \nabla u_\infty) = 0$, so we have

$$
\langle u_\infty, u_n\rangle_\gamma = \int_{\partial R} u_n(\gamma \nabla u_\infty \cdot \vec{n}) ds
$$

Note that $u_n|_{\partial R} = \phi_n \to \phi = u_\infty|_{\partial R}$ uniformly, so, reversing integration by parts and reusing γ-harmonicity of u_∞:

$$
\begin{aligned}
\lim_{n\to\infty} \langle u_n, u_\infty\rangle_\gamma &= \int_{\partial R} \lim_{n\to\infty} (u_n(\gamma \nabla u_\infty \cdot \vec{n})) ds \\
&= \int_{\partial R} u_\infty(\gamma \nabla u_\infty \cdot \vec{n}) ds \\
&= \int_{\partial R} u_\infty(\gamma \nabla u_\infty \cdot \vec{n}) ds - \int_R u_\infty \nabla(\gamma \nabla u_\infty) \mathrm{d}A \\
&= \int_R \gamma |\nabla u_\infty|^2 \mathrm{d}A \\
&= \|u_\infty\|_\gamma^2
\end{aligned}
$$

It remains to be shown that

$$\lim_{n\to\infty} \|u_n\|_\gamma = \|u_\infty\|_\gamma.$$

Define the following sequence of semi-norms of $u\colon R \to \mathbb{R}$:

$$\|u\|^2_{\gamma,n} = \sum_{i=1}^{2n^2} \gamma_i(\max_{T_i} |u_x|^2 + \max_{T_i} |u_y|^2)|T_i|$$

Note that if $u \in S_n$, since $|u_x|$ and $|u_y|$ are constant over each T_i,

$$\begin{aligned}
\|u\|^2_{\gamma,n} &= \sum_{i=1}^{2n^2} \gamma_i |\nabla_{T_i} u|^2 |T_i| \\
&= \sum_{i=1}^{2n^2} |\nabla_{T_i} u|^2 \int_{T_i} \gamma \mathrm{d}A \\
&= \int_R \gamma |\nabla u|^2 \mathrm{d}A \\
&= \|u\|^2_\gamma
\end{aligned}$$

Consider the minimizer, v_n, of these norms over all functions $u\colon R \to \mathbb{R}$ such that $u(x,y) = \phi_n(x,y)$ for all (x,y) which are intersection points of the n by n grid on the boundary.

Given one of these functions $u\colon R \to \mathbb{R}$, create $\bar{u} \in S_n$ by defining $\bar{u}(x,y) = u(x,y)$ for (x,y) intersection points on the n by n grid, then linearly interpolating through the rectangle.

Note that $\max_{T_i} \bar{u}_x \le \max_{T_i} u_x$. One leg of T_i is parallel to the x-axis (call this leg $T_{i,x}$), and $\bar{u}$ and u have the same values at either end, by construction of $\bar{u}$. Hence by the single variable mean value theorem, there is $\xi \in T_{i,x}$ such that $u_x(\xi) = (u(b) - u(a))/|T_{i,x}| = \bar{u}_x(\zeta)$ for all $\zeta \in T_{i,z}$, since $\bar{u}_x$ is constant over $T_{i,x}$, where a and b are the endpoints of $T_{i,x}$. Hence, $\max_{T_i} u_x \ge \bar{u}_x = \max_{T_i} \bar{u}_x$. Likewise, since one leg of T_i is parallel to the y-axis, $\max_{T_i} u_y \ge \bar{u}_y$, whereby we conclude

$$\begin{aligned}
\|u\|_{\gamma,n} &= \sum_{i=1}^{2n^2} \gamma_i (\max_{T_i} |u_x|^2 + \max_{T_i} |u_y|^2)|T_i| \\
&\geq \sum_{i=1}^{2n^2} \gamma_i (|u_x|_{T_i}^2 + |u_y|_{T_i}^2)|T_i| \\
&= \|\bar{u}\|_{\gamma,n} \\
&= \|\bar{u}\|_{\gamma}
\end{aligned}$$

Since $\bar{u} \in S_n$ and $\bar{u}|_{\partial R} = \phi_n$,

$$\|u\|_{\gamma,n} \geq \|\bar{u}\|_{\gamma,n} = \|\bar{u}\|_{\gamma} \geq \|u_n\|_{\gamma} = \|u_n\|_{\gamma,n}.$$

So $u_n = v_n$. Thus, since $u_\infty|_{\partial R} = \phi$, and therefore $u_\infty(x,y) = \phi_n(x,y)$ for all (x,y) whcich are intersection points of the n by n grid on the boundary, we have that

$$\|u_\infty\|_{\gamma,n} \geq \|u_n\|_{\gamma,n} = \|u_n\|_{\gamma}.$$

Furthermore,

$$\lim_{n\to\infty} \|u\|_{\gamma,n} = \|u\|_{\gamma}$$

as $\|u\|_{\gamma,n}$ is simply a Riemann sum of $\|u\|_{\gamma}$. Hence,

$$\lim_{n\to\infty} \|u_\infty\|_{\gamma,n} = \|u_\infty\|_{\gamma} \geq \limsup_{n\to\infty} \|u_n\|_{\gamma}.$$

There exists a heuristic argument for the fact that $\|u_\infty\| \leq \liminf_{n\to\infty} \|u_n\|_{\gamma}$, however, a complete proof eludes us.